Derivatives of the Earth's potentials

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ABSTRACT

The real Earth, and its theoretical model, the normal ellipsoid, have three *potentials* of interest in geodesy. They are the Earth's gravity potential W, the normal gravity potential U and the disturbing potential $T = W - U$.

The potential W can be approximated by a geopotential model (a series of spherical harmonic terms and potential coefficients) plus the rotational potential of the Earth. The potential of the normal ellipsoid U (which can be computed exactly from its defining parameters) and the disturbing potential T are also conveniently represented by spherical harmonic series.

Derivatives of potentials W, U and T are required for a variety of geodetic purposes, such as the computation of gravity anomalies and deflections of the vertical, as well as the calculation of coefficients in the unified geodetic adjustment process known as *collocation*. This paper describes a summation method, based on *Clenshaw's recurrence formula*, which is an efficient numerical technique for computing potentials and their derivatives.

INTRODUCTION

Clenshaw's recurrence formula (Clenshaw 1955) and the associated summation algorithm (hereafter combined and called *Clenshaw's summation*) is a fast and numerically stable technique for evaluating series of polynomials which satisfy recurrence formulae (Press *et al.* 1992, pp. 178-183). In geodesy, gravity potentials of the Earth W, the normal ellipsoid U and the disturbing potential $T = W - U$ are conveniently expressed as spherical harmonic series containing associated Legendre functions. Associated Legendre functions are orthogonal polynomials that satisfy well-known recurrence relationships, hence series containing them are eminently suitable for evaluation using Clenshaw's summation. In addition, Clenshaw's summation can be extended to include the numerical computation of first and second-order derivatives of W, U and T with respect to polar coordinates r,ψ,λ or r,t,λ (r is geocentric radial distance, $t = sin\psi$ where ψ is geocentric latitude and λ is longitude).

In this paper, expressions for the potentials W, U and T are developed in suitable forms, recurrence relationships for associated Legendre functions set down and Clenshaw's summation explained and developed to include the calculation of first and second-order derivatives. These derivatives (with respect to r,t,λ) can be transformed to derivatives with respect to *geocentric* Cartesian coordinates X,Y,Z or *local* Cartesian coordinates x,y,z by techniques set out by Tscherning (1976a). This paper provides a re-statement of these methods and formulae.

Clenshaw's elegant numerical method as applied to geodetic problems and the transformation of derivatives from one orthogonal coordinate system to another have been well documented by Tscherning (1976a, 1976b), Tscherning and Poder (1981), Tscherning, Rapp and Goad (1983) and Gleeson (1985). This paper does not present original development but it is hoped that the setting and presentation will add to the body of knowledge on this topic.

An appendix to this paper contains a computer algorithm for the calculation of the Earth's gravitational potential.

THE EARTH'S GRAVITY POTENTIAL

The gravity potential of the Earth $W(r,\psi,\lambda)$ is the sum of its gravitational potential V_W and rotational potential R

$$W(r,\psi,\lambda) = V_W(r,\psi,\lambda) + R(r,\psi) \tag{1}$$

where the rotational potential of the Earth is (Tscherning 1976b, p. 126, eq. 3)

$$R(r,\psi) = \frac{\omega^2}{2}(r\cos\psi)^2 \tag{2}$$

and ω is the angular velocity of the Earth.

An approximation of V_W is represented as a series of spherical harmonics and coefficients in two well known forms, (a) Tscherning and Poder (1981, p. 270, eq. 26)

$$V_W(r,\psi,\lambda) = \frac{GM}{r}\sum_{n=0}^{N}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right)P_n^m(t) \tag{3a}$$

and (b) Tscherning, Rapp and Goad (1983, p. 250, eq. 1)

$$V_W(r,\psi,\lambda) = \frac{GM}{r}\left[1 + \sum_{n=2}^{N}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right)P_n^m(t)\right] \tag{3b}$$

where r,ψ,λ are polar coordinates (r geocentric radial distance, ψ geocentric latitude and λ longitude), $t = \sin\psi$, GM is the product of the Earth's gravitational constant G and its mass M, a is the semi-major axis of the reference ellipsoid, n and m are positive integers or zero, C_n^m and S_n^m are *geopotential coefficients* of nth degree and mth order, $P_n^m(t)$ are *associated Legendre functions* and N is the maximum degree and order of the available coefficients.

Both equations represent the same geopotential model of the Earth's gravitational potential, (3b) differing from (3a) only in the starting value of n.

This is because (i) the first term of an harmonic series ($n = 0$) always represents the mean value of the function, in this case GM/r; and (ii) coefficients C_1^0, C_1^1 and S_1^1 are set to zero to force the coordinate origin to coincide with the Earth's centre of mass [coefficients S_n^0 will always be zero since when $m=0$, $\sin(m\lambda) = 0$ and if equation (3a) is used then $C_0^0 = 1$].

Another form of the potential can be developed from equation (3a) by letting $q = a/r$ and re-arranging the double summation as (Tscherning, Rapp & Goad 1983, p. 255, eq. 10)

$$V_W(r,\psi,\lambda) = \frac{GM}{r} \sum_{m=0}^{N} \sum_{n=m}^{N} \left(C_n^m \cos m\lambda + S_n^m \sin m\lambda \right) q^n P_n^m(t) \tag{4}$$

where $C_0^0 = 1$ and C_1^0, C_1^1 and S_1^1 are zero.

Equations (3a), (3b) and (4) are given with *conventional* (or un-normalised) coefficients and associated Legendre functions, but in practice, *fully-normalised* and *quasi-normalised* coefficients and associated Legendre functions are used. In this paper fully-normalised terms are denoted as $\overline{C}_n^m$, $\overline{S}_n^m$, $\overline{P}_n^m(t)$ and quasi-normalised terms as $\widetilde{C}_n^m$, $\widetilde{S}_n^m$, $\widetilde{P}_n^m(t)$.

Fully-normalised, quasi-normalised and conventional associated Legendre functions and coefficients are related in the following way (Heiskanen & Moritz 1967, p. 32 and Tscherning, Rapp & Goad 1983, eqs 20 & 21, p. 258)

$$\overline{P}_n^m(t) = \sqrt{\frac{k(2n+1)(n-m)!}{(n+m)!}}\, P_n^m(t) \qquad \begin{Bmatrix} \overline{C}_n^m \\ \overline{S}_n^m \end{Bmatrix} = \sqrt{\frac{(n+m)!}{k(2n+1)(n-m)!}} \begin{Bmatrix} C_n^m \\ S_n^m \end{Bmatrix} \tag{5a}$$

where $k = 1$ when $m = 0$
$k = 2$ when $m > 0$

$$\widetilde{P}_n^m(t) = \sqrt{\frac{(n-m)!}{(n+m)!}}\, P_n^m(t) \qquad \begin{Bmatrix} \widetilde{C}_n^m \\ \widetilde{S}_n^m \end{Bmatrix} = \sqrt{\frac{(n+m)!}{(n-m)!}} \begin{Bmatrix} C_n^m \\ S_n^m \end{Bmatrix} \tag{5b}$$

Using equations (5a) and (5b) the relationship between quasi-normalised and fully-normalised potential coefficients may be given as

$$\begin{Bmatrix} \widetilde{C}_n^m \\ \widetilde{S}_n^m \end{Bmatrix} = \sqrt{k(2n+1)} \begin{Bmatrix} \overline{C}_n^m \\ \overline{S}_n^m \end{Bmatrix} \tag{5c}$$

where $k = 1$ when $m = 0$
$k = 2$ when $m > 0$

THE NORMAL GRAVITY POTENTIAL

The *normal ellipsoid* is an homogeneous ellipsoid concentric with and having the same mass as the Earth, and rotating with the same angular velocity about the Earth's axis. It generates a theoretical gravity field known as the *normal gravity field*, and its surface is an equipotential surface of the normal gravity field. The normal ellipsoid is a reference surface for both position and gravity and its currently accepted parameters are defined by the Geodetic Reference System 1980 (GRS80) [BG 1988, p. 348] as

- equatorial radius of the Earth
  $a = 6378137 m$
- geocentric gravitational constant of the Earth (including the atmosphere)
  $GM = 3986005 \times 10^8 \, m^3/s^2$
- dynamical form factor of the Earth (excluding the permanent tidal deformation)
  $J_2 = 108263 \times 10^{-8}$
- angular velocity of the Earth
  $\omega = 7292115 \times 10^{-11} \, radians/s$

Similarly to the Earth, the gravity potential of the normal ellipsoid U is the sum of its gravitational and rotational potentials (except that U is independent of longitude since the normal ellipsoid is a rotationally symmetric body)

$$U(r,\psi) = V_U(r,\psi) + R(r,\psi) \tag{6}$$

Its rotational potential is given by equation (2) and its gravitational potential V_U can be computed from the series of spherical harmonics and potential coefficients (Heiskanen & Moritz 1967, p. 73, eq. 2-92)

$$V_U(r,\psi) = \frac{GM}{r}\left[1 - \sum_{n=1}^{\infty}\left\{\left(\frac{a}{r}\right)^{2n} J_{2n}\, P_{2n}(t)\right\}\right] \tag{7}$$

where $P_{2n}(t)$ are *Legendre polynomials* and J_{2n} are *normal potential coefficients* derived from the dynamical form factor J_2 by

$$J_{2n} = (-1)^{n+1}\frac{3e^{2n}}{(2n+1)(2n+3)}\left(1-n+5n\frac{J_2}{e^2}\right) \tag{8}$$

and e is the eccentricity of the normal ellipsoid, computed by iterative techniques using the defining parameters of the normal ellipsoid (BG 1988, p. 351).

Another form of equation (7), suitable for evaluation by Clenshaw's summation, is obtained by letting $q = a/r$, noting that $P_n^0(t) = P_n(t)$ and only even coefficients $J_2, J_4, J_6, \cdots$ *etc* are required. Then with a set of coefficients A_n^0, defined as

$$A_0^0 = 1,\ A_1^0 = 0,\ A_2^0 = -J_2,\ A_3^0 = 0,\ A_4^0 = -J_4,\ A_5^0 = 0,\ A_6^0 = -J_6, \cdots\ etc$$

and limiting the summation to N terms, the normal gravitational potential is given as

$$V_U(r,\psi) = \frac{GM}{r}\sum_{n=0}^{N} A_n^0\, q^n\, P_n^0(t) \tag{9}$$

For the purpose of developing an expression for the *disturbing potential T*, equation (9) can be given in the apparently trivial form

$$V_U(r,\psi) = \frac{GM}{r}\sum_{m=0}^{N}\sum_{n=m}^{N}\left(A_n^m \cos m\lambda + B_n^m \sin m\lambda\right) q^n\, P_n^m(t) \tag{10}$$

where <u>all</u> $B_n^m = 0$, <u>all</u> $A_n^m = 0$ when $m > 0$, $P_n^0(t) = P_n(t)$ and the coefficients A_n^0 are

$$A_0^0 = 1,\ A_1^0 = 0,\ A_2^0 = -J_2,\ A_3^0 = 0,\ A_4^0 = -J_4,\ A_5^0 = 0,\ A_6^0 = -J_6, \cdots\ etc$$

THE DISTURBING POTENTIAL

The *disturbing potential* $T = W - U$ is equivalent to $V_W - V_U$ (since the rotational potentials of the Earth and the normal ellipsoid cancel in the subtraction); hence from equations (4) and (10), remembering that all $B_n^m = 0$, a spherical harmonic series for T is given as

$$T(r,\psi,\lambda) = \frac{GM}{r}\sum_{m=0}^{N}\sum_{n=m}^{N}\left\{\left(C_n^m - A_n^m\right)\cos m\lambda + S_n^m \sin m\lambda\right\} q^n P_n^m(t) \tag{11}$$

where $C_0^0 = 1$ and $C_1^0 = C_1^1 = S_1^1 = 0$,
all $A_n^m = 0$ when $m > 0$, and when $m = 0$ the coefficients A_n^0 are

$$A_0^0 = 1,\ A_1^0 = 0,\ A_2^0 = -J_2,\ A_3^0 = 0,\ A_4^0 = -J_4,\ A_5^0 = 0,\ A_6^0 = -J_6, \cdots \ etc$$

LEGENDRE FUNCTIONS AND RECURRENCE FORMULAE

Associated Legendre functions $P_n^m(t)$ and Legendre polynomials $P_n(t)$ are related by

$$P_n^m(t) = u^m \frac{d^m}{dt^m} P_n(t) \qquad \text{and} \qquad P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2 - 1\right)^n \tag{12}$$

where, by definition: $P_n^m(t) = 0$ if $n < 0$ or $m > n$, $P_n^0(t) = P_n(t)$,
$t = \sin\psi$ and $u = \sqrt{1 - t^2} = \cos\psi$.

These equations (Heiskanen & Moritz 1967, pp. 22-23) are used to calculate "special" values: $P_0^0(t) = 1$, $P_1^0(t) = t$, $P_1^1(t) = u$, but are unsuitable for calculating higher degree and order values of $P_n^m(t)$; such values are computed from *recurrence formulae* for which the special values act as "seeds". Two of these well-known formulae are (Gleeson 1985, p. 116, eqs 2.1 & 2.2)

$$P_n^m(t) - \frac{2n-1}{n-m} t P_{n-1}^m(t) + \frac{n+m-1}{n-m} P_{n-2}^m(t) = 0 \tag{13}$$

$$P_m^m(t) - (2m-1) u P_{m-1}^{m-1}(t) = 0 \tag{14}$$

Letting $n = m + 1$ in equation (13), noting that $P_n^m(t)=0$ if $m > n$, gives a special result

$$P_{m+1}^m(t) - (2m+1)tP_m^m(t) = 0 \tag{15}$$

Equations (13) and (15) can be written in a general polynomial form as

$$P_n^m(t) + a_n^m(t)P_{n-1}^m(t) + b_n^m P_{n-2}^m(t) = 0 \tag{16}$$

and

$$P_{m+1}^m(t) + a_{m+1}^m(t)P_m^m(t) = 0 \tag{17}$$

where the polynomial coefficients $a_n^m(t)$ and b_n^m are

$$a_n^m(t) = -\frac{2n-1}{n-m}t \qquad \text{and} \qquad b_n^m = \frac{n+m-1}{n-m} \tag{18}$$

noting that $a_n^m(t)$ is a linear function of t and b_n^m is independent of t.

For values of m such that $0 \le m \le N$, the recurrence formulae [eqs (16) & (17)] can be represented in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ a_{m+1}^m(t) & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ b_{m+2}^m & a_{m+2}^m(t) & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & b_{m+3}^m & a_{m+3}^m(t) & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \ddots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & b_{N-1}^m & a_{N-1}^m(t) & 1 & 0 \\ 0 & 0 & \cdot & \cdot & 0 & b_N^m & a_N^m(t) & 1 \end{bmatrix} \begin{bmatrix} P_m^m(t) \\ P_{m+1}^m(t) \\ P_{m+2}^m(t) \\ P_{m+3}^m(t) \\ \cdot \\ \cdot \\ P_{N-1}^m(t) \\ P_N^m(t) \end{bmatrix} = \begin{bmatrix} P_m^m(t) \\ 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

or $$\boldsymbol{A}\boldsymbol{p} = \boldsymbol{p}_0 \tag{19}$$

where $\boldsymbol{A}$ is a coefficient matrix (whose inverse $\boldsymbol{A}^{-1}$ exists since the determinant of $\boldsymbol{A}$ is 1), $\boldsymbol{p}$ is a vector of associated Legendre functions and $\boldsymbol{p}_0$ is a vector containing only $P_m^m(t)$.

The recurrence relationships above, with polynomial coefficients $a_n^m(t)$ and b_n^m, are given for conventional associated Legendre functions $P_n^m(t)$. Similar formulae may be derived for quasi- and fully-normalised functions, $\widetilde{P}_n^m(t)$ and $\overline{P}_n^m(t)$ respectively, by substituting the relationships contained in equations (5a) and (5b) into equations (13) and (14) to give

$$\widetilde{P}_n^m(t) - \frac{2n-1}{\sqrt{(n+m)(n-m)}}\, t\, \widetilde{P}_{n-1}^m(t) + \sqrt{\frac{(n+m-1)(n-m-1)}{(n+m)(n-m)}}\, \widetilde{P}_{n-2}^m(t) = 0 \tag{20}$$

$$\widetilde{P}_m^m(t) - \sqrt{\frac{2m-1}{2m}}\, u\, \widetilde{P}_{m-1}^{m-1}(t) = 0 \tag{21}$$

with special values $\widetilde{P}_0^0(t)=1$, $\widetilde{P}_1^0(t)=t$ and $\widetilde{P}_1^1(t)=\frac{u}{\sqrt{2}}$, and

$$\overline{P}_n^m(t) - \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}\, t\, \overline{P}_{n-1}^m(t) + \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}}\, \overline{P}_{n-2}^m(t) = 0 \tag{22}$$

$$\overline{P}_m^m(t) - \sqrt{\frac{2m+1}{2m}}\, u\, \overline{P}_{m-1}^{m-1}(t) = 0 \tag{23}$$

with special values $\overline{P}_0^0(t)=1$, $\overline{P}_1^0(t)=\sqrt{3}\,t$ and $\overline{P}_1^1(t)=\sqrt{3}\,u$

Another set of recurrence relationships for *modified Legendre functions* is useful when it is realised that the equations for the potentials [eqs (4), (9) & (11)] are simplified by defining the *modified* conventional Legendre function $p_n^m(t)$ as

$$p_n^m(t) = q^n P_n^m(t) \tag{24}$$

Recurrence formulae for these functions can be obtained by first re-ordering equation (24) as $P_n^m(t)=q^{-n}p_n^m(t)$, then letting $n=n-1$ and $n=n-2$ to give $P_{n-1}^m(t)=q^{-n+1}p_{n-1}^m(t)$ and $P_{n-2}^m(t)=q^{-n+2}p_{n-2}^m(t)$. Also, with $n = m$, $P_m^m(t)=q^{-m}p_m^m(t)$, and letting $m=m-1$ gives $P_{m-1}^{m-1}(t)=q^{-m+1}p_{m-1}^{m-1}(t)$. Substituting these expressions into equations (13) and (14) gives

$$p_n^m(t) - \frac{2n-1}{n-m}\, t\, q\, p_{n-1}^m(t) + \frac{n+m-1}{n-m}\, q^2\, p_{n-2}^m(t) = 0 \tag{25}$$

$$p_m^m(t) - (2m-1)\, u\, q\, p_{m-1}^{m-1}(t) = 0 \tag{26}$$

Modified Legendre functions can also be expressed in quasi- or fully-normalised form as

$$\tilde{p}_n^m(t) = q^n \tilde{P}_n^m(t) \qquad \text{or} \qquad \bar{p}_n^m(t) = q^n \bar{P}_n^m(t) \tag{27}$$

and using similar substitutions as above into equations (20)-(23) gives recurrence formulae for quasi- and fully-normalised modified Legendre functions as

$$\tilde{p}_n^m(t) - \frac{2n-1}{\sqrt{(n+m)(n-m)}}\, t\, q\, \tilde{p}_{n-1}^m(t) + \sqrt{\frac{(n+m-1)(n-m-1)}{(n+m)(n-m)}}\, q^2\, \tilde{p}_{n-2}^m(t) = 0 \tag{28}$$

$$\tilde{p}_m^m(t) - \sqrt{\frac{2m-1}{2m}}\, u\, q\, \tilde{p}_{m-1}^{m-1}(t) = 0 \tag{29}$$

$$\bar{p}_n^m(t) - \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}\, t q \bar{p}_{n-1}^m(t) + \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}}\, q^2\, \bar{p}_{n-2}^m(t) = 0 \tag{30}$$

$$\bar{p}_m^m(t) - \sqrt{\frac{2m+1}{2m}}\, u\, q\, \bar{p}_{m-1}^{m-1}(t) = 0 \tag{31}$$

Three sets of recurrence relationships for Legendre functions are shown above:

(i) conventional [eqs (13) & (14)] and modified conventional [eqs (25) & (26)],

(ii) quasi-normalised [eqs (20) & (21)] and modified quasi-normalised [eqs (28) & (29)], and

(iii) fully-normalised [eqs (22) & (23)] and modified fully-normalised [eqs (30) & 31)].

In the recurrence relationships, the Legendre functions have polynomial coefficients $a_n^m(t)$ and b_n^m which take different forms depending on the formulae used, noting that in the second equation of each pair $u = \sqrt{1-t^2}$ and $b_n^m = 0$.

CLENSHAW'S SUMMATION TECHNIQUE

Equations (4), (9) and (11) are the fundamental equations for the potentials V_W, V_U and T [shown with conventional terms C_n^m, S_n^m, A_n^m and $P_n^m(t)$ but also expressible with fully- or quasi-normalised terms, $\overline{C}_n^m$, $\overline{S}_n^m$, $\overline{A}_n^m$, $\overline{P}_n^m(t)$ and $\widetilde{C}_n^m$, $\widetilde{S}_n^m$, $\widetilde{A}_n^m$, $\widetilde{P}_n^m(t)$, respectively]. Each equation has a common "kernel", a summation term v^m having the general form

$$v^m = \sum_{n=m}^{N} y_n^m \, p_n^m(t) = y^T p \tag{32}$$

where y is a vector containing the y_n^m coefficients (y^T is the transpose of y) and p is a vector containing the conventional modified Legendre functions $p_n^m(t) = q^n P_n^m(t)$.

Now, from matrix equation (19) $p = A^{-1} p_0$ and equation (32) becomes

$$v^m = \sum_{n=m}^{N} y_n^m \, p_n^m(t) = y^T p = y^T A^{-1} p_0 \tag{33}$$

which, since v^m is a scalar quantity (equal to its transpose) and $(A^{-1})^T = (A^T)^{-1}$, can be written as

$$v^m = (y^T A^{-1} p_0)^T = p_0^T (A^{-1})^T y = p_0^T (A^T)^{-1} y \tag{34}$$

The pattern of the elements in $(A^T)^{-1}$, due to the upper-triangular form of A^T arising from the recurrence relationships, enables the summation v^m to be calculated in the following way.

Let s be a vector

$$s = \begin{bmatrix} s_m^m \\ s_{m+1}^m \\ s_{m+2}^m \\ \vdots \\ s_N^m \end{bmatrix} = (A^T)^{-1} y \tag{35}$$

then with $s_{N+2}^m = s_{N+1}^m = 0$, the elements of s can be calculated recursively from

$$s_n^m = -a_{n+1}^m(t) s_{n+1}^m - b_{n+2}^m s_{n+2}^m + y_n^m \quad \text{where} \quad m \le n \le N \tag{36}$$

Equation (34) now becomes

$$v^m = \sum_{n=m}^{N} y_n^m p_n^m(t) = p_0^T s = \begin{bmatrix} p_m^m(t) & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} s_m^m \\ s_{m+1}^m \\ s_{m+2}^m \\ \vdots \\ s_N^m \end{bmatrix}$$

or simply

$$v^m = \sum_{n=m}^{N} y_n^m p_n^m(t) = s_m^m p_m^m(t) \tag{37}$$

Thus, the summation of the products of coefficients y_n^m and the Legendre functions $p_n^m(t)$ is simply the product of a single number s_m^m, calculated recursively from a sequence of coefficients $a_n^m(t)$, b_n^m and y_n^m, and the single Legendre function $p_m^m(t)$. When calculating potentials, Clenshaw's summation is even more simple, since summations begin at $m = N$ and end at $m = 0$, hence $p_0^0(t) = 1$ and the potential is just the number s_0^0.

EVALUATION OF THE EARTH'S POTENTIAL USING CLENSHAW'S SUMMATION

The Earth's gravitational potential, in quasi-normalised form, can be written as

$$V_W(r,\psi,\lambda) = \frac{GM}{r}\sum_{m=0}^{N}\left\{(\cos m\lambda)\,\tilde{v}_c^m + (\sin m\lambda)\,\tilde{v}_s^m\right\} \tag{38}$$

with the summation terms

$$\tilde{v}_c^m = \sum_{n=m}^{N}\tilde{C}_n^m\,\tilde{p}_n^m(t) = \tilde{s}_{m_c}^m\,\tilde{p}_m^m(t) \tag{39a}$$

$$\tilde{v}_s^m = \sum_{n=m}^{N}\tilde{S}_n^m\,\tilde{p}_n^m(t) = \tilde{s}_{m_s}^m\,\tilde{p}_m^m(t) \tag{39b}$$

where subscripts and sub-subscripts c and s refer to the potential coefficients $\tilde{C}_n^m$ and $\tilde{S}_n^m$.

The elements $\tilde{s}_{m_c}^m$ and $\tilde{s}_{m_s}^m$ are found by using Clenshaw's recursive summation [eq. (36)] with polynomial coefficients $\tilde{a}_{n+1}^m(t)$ and $\tilde{b}_{n+2}^m$ (obtained from eq. (28) with $n = n + 1$ and $n = n + 2$) as

$$\tilde{a}_{n+1}^m(t) = -\frac{2n+1}{\sqrt{(n+m+1)(n-m+1)}}\,t\,q \qquad \tilde{b}_{n+2}^m = \sqrt{\frac{(n+m+1)(n-m+1)}{(n+m+2)(n-m+2)}}\,q^2$$

and the coefficients $\tilde{y}_n^m = \tilde{C}_n^m$ and $\tilde{S}_n^m$

Substituting equations (39a) and (39b) into (38) gives

$$V_W(r,\psi,\lambda) = \frac{GM}{r}\sum_{m=0}^{N}\left\{(\cos m\lambda)\,\tilde{s}_{m_c}^m + (\sin m\lambda)\,\tilde{s}_{m_s}^m\right\}\tilde{p}_m^m(t) \tag{40}$$

where the potential V_W can be evaluated by a second implementation of Clenshaw's recursive summation [eq. (36)], this time using equation (29) with $m = m + 1$ giving

$$\tilde{a}^m_{m+1}(t) = -\sqrt{\frac{2m+1}{2m+2}}\, u\, q \qquad \tilde{b}^m_{m+2} = 0$$

and the coefficient $\tilde{y}^m_m = (\cos m\lambda)\, \tilde{s}^m_{m_c} + (\sin m\lambda)\, \tilde{s}^m_{m_s}$

Appendix A contains a computer algorithm, in the C language, for calculating the Earth's gravitational potential V_W. It contains an outer loop $(m = N$ to $m = 0)$ and an inner loop $(n = N$ to $n = m)$, both loops decrementing by one after each pass. The sums $\tilde{s}^m_{m_c}$ and $\tilde{s}^m_{m_s}$ (shown as `sc` and `ss` in the computer code) are calculated in the inner loop by a first application of Clenshaw's summation with polynomial coefficients $\tilde{a}^m_{n+1}(t)$ and $\tilde{b}^m_{n+2}$ (`a1`, `b2`) and coefficients $\tilde{y}^m_n$ equal to $\tilde{C}^m_n$ or $\tilde{S}^m_n$ (`Cnm`, `Snm`). These sums are then used in the outer loop to calculate the potential V_W (`V`) in a second application of Clenshaw's summation with a new polynomial coefficient $\tilde{a}^m_{m+1}(t)$ and coefficient $\tilde{y}^m_n = (\cos m\lambda)\, \tilde{s}^m_{m_c} + (\sin m\lambda)\, \tilde{s}^m_{m_s}$. <u>Note that no Legendre functions are evaluated</u>.

Appendix A also contains a "result" for the Earth's potentials using a set of quasi-normalised geopotential coefficients of degree and order four and some constants of the Geodetic Reference System 1980 (GRS80). Geopotential models usually contain fully-normalised coefficients $\bar{C}^m_n$ and $\bar{S}^m_n$ but small computational savings can be made in computer algorithms by using quasi-normalised coefficients $\tilde{C}^m_n$, $\tilde{S}^m_n$ and recursive formulae whose coefficients $\tilde{a}^m_n(t)$ and $\tilde{b}^m_n$ are more simple than their fully-normalised equivalents $\bar{a}^m_n(t)$ and $\bar{b}^m_n$ [see eqs (28) & (30)]. In this event, $\bar{C}^m_n$ and $\bar{S}^m_n$ must be converted to $\tilde{C}^m_n$ and $\tilde{S}^m_n$ by using equation (5c).

DERIVATIVES OF THE POTENTIALS WITH RESPECT TO POLAR COORDINATES r,ψ,λ

For various geodetic purposes partial derivatives of the potentials W, U and T are required, such potentials being functions of the variables r,ψ,λ or r,t,λ where $t = \sin\psi$.

In the case of the Earth's gravity potential $W(r,t,\lambda) = V_W(r,t,\lambda) + R(r,t)$ the partial derivatives of interest are (dropping the subscript from the gravitational potential V_W)

$$\frac{\partial W}{\partial r} = \frac{\partial V}{\partial r} + \frac{\partial R}{\partial r}, \qquad \frac{\partial W}{\partial t} = \frac{\partial V}{\partial t} + \frac{\partial R}{\partial t}, \qquad \frac{\partial W}{\partial \lambda} = \frac{\partial V}{\partial \lambda} \tag{41a}$$

$$\frac{\partial^2 W}{\partial r \partial t} = \frac{\partial^2 V}{\partial r \partial t} + \frac{\partial^2 R}{\partial r \partial t}, \qquad \frac{\partial^2 W}{\partial r \partial \lambda} = \frac{\partial^2 V}{\partial r \partial \lambda}, \qquad \frac{\partial^2 W}{\partial t \partial \lambda} = \frac{\partial^2 V}{\partial t \partial \lambda} \tag{41b}$$

$$\frac{\partial^2 W}{\partial r^2} = \frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 R}{\partial r^2}, \qquad \frac{\partial^2 W}{\partial t^2} = \frac{\partial^2 V}{\partial t^2} + \frac{\partial^2 R}{\partial t^2}, \qquad \frac{\partial^2 W}{\partial \lambda^2} = \frac{\partial^2 V}{\partial \lambda^2} \tag{41c}$$

[Note that the chain-rule for differentiation can be used to convert derivatives with respect to t to derivatives with respect to ψ, for example: $\dfrac{\partial V}{\partial \psi} = \dfrac{\partial V}{\partial t}\dfrac{\partial t}{\partial \psi}$ where $\dfrac{\partial t}{\partial \psi} = \cos\psi = u$]

Similar derivatives of the potential U can be written by replacing W with U in equations (41) noting that in this case V would be V_U and there would be no derivatives with respect to λ since the equipotential ellipsoid is rotationally symmetric. Partial derivatives of the disturbing potential T are also often required, mainly for the calculation of gravity anomalies and deflections of the vertical, they are $\dfrac{\partial T}{\partial r}$, $\dfrac{\partial T}{\partial \psi}$ and $\dfrac{\partial T}{\partial \lambda}$.

In this paper, only derivatives of the Earth's gravity potential W will be developed fully but the processes can be applied to derivatives of U and T which in many cases are more simple expressions.

The following partial derivatives of the rotational potential R are obtained by differentiating equation (2) remembering $t = \sin\psi$ and $u = \sqrt{1-t^2} = \cos\psi$

$$\frac{\partial R}{\partial r} = \omega^2 r \cos^2 \psi = \omega^2 r u^2 \tag{42}$$

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial \psi}\frac{\partial \psi}{\partial t} = -\omega^2 r^2 \cos\psi \sin\psi \frac{1}{\cos\psi} = -\omega^2 r^2 t \tag{43}$$

$$\frac{\partial^2 R}{\partial r \partial t} = -2\omega^2 r t \tag{44}$$

$$\frac{\partial^2 R}{\partial r^2} = \omega^2 u^2 \tag{45}$$

$$\frac{\partial^2 R}{\partial t^2} = -\omega^2 r^2 \tag{46}$$

These derivatives, added to those of the Earth's gravitational potential V_W (hereafter expressed simply as V) give derivatives of W according to equations (41).

Partial derivatives of V can be divided into two groups, (i) derivatives with respect to r and λ and (ii) derivatives with respect to t. The techniques of solution are slightly different for each group.

Derivatives of V with respect to r and λ

The following derivatives (Hopkins 1973, pp. 26-27) are obtained by partially differentiating equation (4), noting that the denominator r can be taken inside the summations, $q = a/r$, $q^n/r = a^n r^{-(n+1)}$ and the modified Legendre function $p_n^m(t) = q^n P_n^m(t)$

$$\frac{\partial V}{\partial r} = -\frac{GM}{r^2}\sum_{m=0}^{N}\sum_{n=m}^{N}(n+1)\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right)p_n^m(t) \tag{47}$$

$$\frac{\partial V}{\partial \lambda} = \frac{GM}{r}\sum_{m=0}^{N}\sum_{n=m}^{N} m\left(S_n^m \cos m\lambda - C_n^m \sin m\lambda\right)p_n^m(t) \tag{48}$$

$$\frac{\partial^2 V}{\partial r \partial \lambda} = \frac{GM}{r^2}\sum_{m=0}^{N}\sum_{n=m}^{N} m(n+1)\left(C_n^m \sin m\lambda - S_n^m \cos m\lambda\right)p_n^m(t) \tag{49}$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{GM}{r^3} \sum_{m=0}^{N} \sum_{n=m}^{N} (n+1)(n+2)\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right) p_n^m(t) \tag{50}$$

$$\frac{\partial^2 V}{\partial \lambda^2} = -\frac{GM}{r} \sum_{m=0}^{N} \sum_{n=m}^{N} m^2 \left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right) p_n^m(t) \tag{51}$$

The derivatives can be re-arranged in forms suitable for Clenshaw's summation noting that each has a summation v^m of the form of equation (37), for example

$$\begin{aligned}
\frac{\partial V}{\partial r} &= \frac{GM}{r^2} \sum_{m=0}^{N} \left\{ \sum_{n=m}^{N} -(n+1)\left(C_n^m \cos m\lambda\right) p_n^m(t) + \sum_{n=m}^{N} -(n+1)\left(S_n^m \sin m\lambda\right) p_n^m(t) \right\} \\
&= \frac{GM}{r^2} \sum_{m=0}^{N} \left\{ \cos m\lambda \sum_{n=m}^{N} -(n+1) C_n^m \, p_n^m(t) + \sin m\lambda \sum_{n=m}^{N} -(n+1) S_n^m \, p_n^m(t) \right\} \\
&= \frac{GM}{r^2} \sum_{m=0}^{N} \left\{ (\cos m\lambda) v_{rc}^m + (\sin m\lambda) v_{rs}^m \right\}
\end{aligned}$$

where
$$v_{rc}^m = \sum_{n=m}^{N} -(n+1) C_n^m \, p_n^m(t) = s_{m_{rc}}^m p_m^m(t)$$
$$v_{rs}^m = \sum_{n=m}^{N} -(n+1) S_n^m \, p_n^m(t) = s_{m_{rs}}^m p_m^m(t)$$

hence

$$\frac{\partial V}{\partial r} = \frac{GM}{r^2} \sum_{m=0}^{N} \left\{ (\cos m\lambda) s_{m_{rc}}^m + (\sin m\lambda) s_{m_{rs}}^m \right\} p_m^m(t) \tag{52}$$

In a similar manner, the remaining derivatives [eqs (48)-(51)] can be written as

$$\begin{aligned}
\frac{\partial V}{\partial \lambda} &= \frac{GM}{r} \sum_{m=0}^{N} m\left\{ (\cos m\lambda) v_s^m - (\sin m\lambda) v_c^m \right\} \\
&= \frac{GM}{r} \sum_{m=0}^{N} m\left\{ (\cos m\lambda) s_{m_s}^m - (\sin m\lambda) s_{m_c}^m \right\} p_m^m(t)
\end{aligned} \tag{53}$$

where
$$v_s^m = \sum_{n=m}^{N} S_n^m \, p_n^m(t) = s_{m_s}^m p_m^m(t)$$
$$v_c^m = \sum_{n=m}^{N} C_n^m \, p_n^m(t) = s_{m_c}^m p_m^m(t)$$

$$\frac{\partial^2 V}{\partial r \partial \lambda} = \frac{GM}{r^2} \sum_{m=0}^{N} m\left\{(\cos m\lambda) v_{rs}^m - (\sin m\lambda) v_{rc}^m\right\}$$

$$\frac{GM}{r^2} \sum_{m=0}^{N} m\left\{(\cos m\lambda) s_{m_{rs}}^m - (\sin m\lambda) s_{m_{rc}}^m\right\} p_m^m(t) \qquad (54)$$

where
$$v_{rc}^m = \sum_{n=m}^{N} -(n+1) C_n^m \, p_n^m(t) = s_{m_{rc}}^m p_m^m(t)$$

$$v_{rs}^m = \sum_{n=m}^{N} -(n+1) S_n^m \, p_n^m(t) = s_{m_{rs}}^m p_m^m(t)$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{GM}{r^3} \sum_{m=0}^{N} \left\{(\cos m\lambda) v_{rrc}^m + (\sin m\lambda) v_{rrs}^m\right\}$$

$$\frac{GM}{r^3} \sum_{m=0}^{N} \left\{(\cos m\lambda) s_{m_{rrc}}^m + (\sin m\lambda) s_{m_{rrs}}^m\right\} p_m^m(t) \qquad (55)$$

where
$$v_{rrc}^m = \sum_{n=m}^{N} (n+1)(n+2) C_n^m \, p_n^m(t) = s_{m_{rrc}}^m p_m^m(t)$$

$$v_{rrs}^m = \sum_{n=m}^{N} (n+1)(n+2) S_n^m \, p_n^m(t) = s_{m_{rrs}}^m p_m^m(t)$$

$$\frac{\partial^2 V}{\partial \lambda^2} = -\frac{GM}{r} \sum_{m=0}^{N} m^2 \left\{(\cos m\lambda) v_c^m + (\sin m\lambda) v_s^m\right\}$$

$$-\frac{GM}{r} \sum_{m=0}^{N} m^2 \left\{(\cos m\lambda) s_{m_c}^m + (\sin m\lambda) s_{m_s}^m\right\} p_m^m(t) \qquad (56)$$

where
$$v_c^m = \sum_{n=m}^{N} C_n^m \, p_n^m(t) = s_{m_c}^m p_m^m(t)$$

$$v_s^m = \sum_{n=m}^{N} S_n^m \, p_n^m(t) = s_{m_s}^m p_m^m(t)$$

The method of calculating the derivatives is identical to that of the potential V, an inner loop where the sums s_m^m are formed using Clenshaw's summation and an outer loop where these sums are combined with other variables in a second application of Clenshaw's summation. Note that no Legendre functions are evaluated.

Derivatives of V with respect to t

The potential V, in quasi-normalised form, was given previously as

$$V = \frac{GM}{r} \sum_{m=0}^{N} \left\{ (\cos m\lambda)\tilde{v}_c^m + (\sin m\lambda)\tilde{v}_s^m \right\} \tag{38}$$

where the summation terms $\tilde{v}_c^m$, $\tilde{v}_s^m$ [defined by equations (39a) and (39b)] are functions of t. Writing derivatives in the form $\frac{\partial y}{\partial x} = \dot{y}$ where appropriate

$$\frac{\partial V}{\partial t} = \frac{GM}{r} \sum_{m=0}^{N} \left\{ (\cos m\lambda)\dot{\tilde{v}}_c^m + (\sin m\lambda)\dot{\tilde{v}}_s^m \right\} \tag{57}$$

The general form of the summations $\tilde{v}_c^m$ and $\tilde{v}_s^m$, given by equation (37) is

$$\tilde{v}^m = \sum_{n=m}^{N} \tilde{y}_n^m \tilde{p}_n^m(t) = \tilde{s}_m^m \tilde{p}_m^m(t) \tag{37}$$

and noting that $\tilde{s}_m^m$ [determined recursively from eq (36) with coefficients $\tilde{a}_n^m(t)$ and $\tilde{b}_n^m$] and $\tilde{p}_n^m(t) = q^n \tilde{P}_n^m(t)$ are both functions of t, the derivative of the sum $\tilde{v}^m$ is

$$\dot{\tilde{v}}^m = \dot{\tilde{s}}_m^m \tilde{p}_m^m(t) + \tilde{s}_m^m \dot{\tilde{p}}_m^m(t) \tag{58}$$

where $\dot{\tilde{p}}_m^m(t) = \frac{\partial}{\partial t}\tilde{p}_m^m(t)$ and $\dot{\tilde{s}}_m^m(t) = \frac{\partial}{\partial t}\tilde{s}_m^m(t)$.

Wang and Guo (1989, eq. 5, p. 240) and Gleeson (1985, eq. 3.15, p. 121) give a recurrence relationship for $\dot{\tilde{p}}_m^m(t)$ which can be written as

$$u^2 \dot{\tilde{p}}_m^m(t) = -mt\tilde{p}_m^m(t) \tag{59}$$

which, when substituted into equation (58) gives the derivative of the summation as

$$\dot{\tilde{v}}^m = \dot{\tilde{s}}_m^m \tilde{p}_m^m(t) - \tilde{s}_m^m \frac{t}{u^2} m \tilde{p}_m^m(t) \qquad (60)$$

Remembering that $\tilde{a}_n^m(t)$ is a linear function of t and $\tilde{b}_n^m$ is independent of t, the elements $\dot{\tilde{s}}_m^m$ can be found by differentiating equation (36) with respect to t, giving the recursive equation (Gleeson 1985, eq. 3.8, p. 120)

$$\dot{\tilde{s}}_n^m = -\tilde{a}_{n+1}^m(t)\dot{\tilde{s}}_{n+1}^m - \tilde{b}_{n+2}^m \dot{\tilde{s}}_{n+2}^m - \dot{\tilde{a}}_{n+1}^m \tilde{s}_{n+1}^m \quad \text{where} \quad m \le n \le N \qquad (61a)$$

with $\dot{\tilde{s}}_{N+2}^m = \dot{\tilde{s}}_{N+1}^m = 0$ and quasi-normalised coefficients

$$\tilde{a}_{n+1}^m(t) = -\frac{2n+1}{\sqrt{(n+m+1)(n-m+1)}}\, t\, q \qquad \tilde{b}_{n+2}^m = \sqrt{\frac{(n+m+1)(n-m+1)}{(n+m+2)(n-m+2)}}\, q^2 \qquad (61b)$$

$$\dot{\tilde{a}}_{n+1}^m = -\frac{2n+1}{\sqrt{(n+m+1)(n-m+1)}}\, q \qquad (61c)$$

Substituting equation (60) into equation (57) and re-arranging gives

$$\frac{\partial V}{\partial t} = \frac{GM}{r}\sum_{m=0}^{N}\left\{\left(\dot{\tilde{s}}_{m_c}^m \cos m\lambda + \dot{\tilde{s}}_{m_s}^m \sin m\lambda\right) - \left(\tilde{s}_{m_c}^m \cos m\lambda + \tilde{s}_{m_s}^m \sin m\lambda\right)\frac{t}{u^2} m\right\}\tilde{p}_m^m(t) \qquad (62)$$

where the sums $\dot{\tilde{s}}_{m_c}^m$, $\dot{\tilde{s}}_{m_s}^m$, $\tilde{s}_{m_c}^m$ and $\tilde{s}_{m_s}^m$ are calculated from a first application of Clenshaw's summation, using equations (61a) and (36) with appropriate coefficients, and the derivative $\frac{\partial V}{\partial t}$ evaluated by a second application of Clenshaw's summation with

$$\tilde{a}_{m+1}^m(t) = -\sqrt{\frac{2m+1}{2m+2}}\, u\, q \qquad \tilde{b}_{m+2}^m = 0$$

and the coefficient $y_m^m = \left(\dot{\tilde{s}}_{m_c}^m \cos m\lambda + \dot{\tilde{s}}_{m_s}^m \sin m\lambda\right) - \left(\tilde{s}_{m_c}^m \cos m\lambda + \tilde{s}_{m_s}^m \sin m\lambda\right)\frac{t}{u^2} m$

Second derivatives $\frac{\partial^2 V}{\partial t \partial r}$ and $\frac{\partial^2 V}{\partial t \partial \lambda}$, found by differentiating equations (52) and (53) with respect to *t*, can be written as

$$\frac{\partial^2 V}{\partial t \partial r} = \frac{GM}{r^2} \sum_{m=0}^{N} \left\{ \left(\dot{\tilde{s}}_{m_{rc}}^{m} \cos m\lambda + \dot{\tilde{s}}_{m_{rs}}^{m} \sin m\lambda \right) - \left(\tilde{s}_{m_{rc}}^{m} \cos m\lambda + \tilde{s}_{m_{rs}}^{m} \sin m\lambda \right) \frac{t}{u^2} m \right\} \tilde{p}_m^m(t) \qquad (63)$$

$$\frac{\partial^2 V}{\partial t \partial \lambda} = \frac{GM}{r} \sum_{m=0}^{N} \left\{ m \left(\dot{\tilde{s}}_{m_s}^{m} \cos m\lambda - \dot{\tilde{s}}_{m_c}^{m} \sin m\lambda \right) - m \left(\tilde{s}_{m_s}^{m} \cos m\lambda - \tilde{s}_{m_c}^{m} \sin m\lambda \right) \frac{t}{u^2} m \right\} \tilde{p}_m^m(t) \qquad (64)$$

Finally, writing second derivatives in the form $\frac{\partial^2 y}{\partial x^2} = \ddot{y}$ where appropriate

$$\frac{\partial^2 V}{\partial t^2} = \frac{GM}{r} \sum_{m=0}^{N} \left\{ (\cos m\lambda)\, \ddot{\tilde{v}}_c^m + (\sin m\lambda)\, \ddot{\tilde{v}}_s^m \right\} \qquad (65)$$

with

$$\ddot{\tilde{v}}^m = \ddot{\tilde{s}}_m^m \tilde{p}_m^m(t) + 2\, \dot{\tilde{s}}_m^m \dot{\tilde{p}}_m^m(t) + \tilde{s}_m^m \ddot{\tilde{p}}_m^m(t) \qquad (66)$$

and the elements $\ddot{\tilde{s}}_m^m$ found from the recursive equation (Gleeson 1985, p. 121)

$$\ddot{\tilde{s}}_n^m = -\tilde{a}_{n+1}^m(t)\, \ddot{\tilde{s}}_{n+1}^m - \tilde{b}_{n+2}^m\, \ddot{\tilde{s}}_{n+2}^m - 2\, \dot{\tilde{a}}_{n+1}^m\, \dot{\tilde{s}}_{n+1}^m \quad \text{where} \quad m \le n \le N \qquad (67)$$

with $\ddot{\tilde{s}}_{N+2}^m = \ddot{\tilde{s}}_{N+1}^m = 0$ and coefficients $\tilde{a}_n^m(t)$, $\tilde{b}_n^m$ and $\dot{\tilde{a}}_n^m$ defined above.

An expression for $\ddot{\tilde{p}}_m^m(t)$, the second derivative of the modified Legendre function, can be found by differentiating equation (59) with respect to *t* and simplifying to give

$$u^4\, \ddot{\tilde{p}}_m^m(t) = \left(\left(m^2 - 1 \right) t^2 - m \right) \tilde{p}_m^m(t) \qquad (68)$$

which, when substituted into equation (66) gives the derivative of the summation as

$$\ddot{\tilde{v}}^m = \ddot{\tilde{s}}_m^m \tilde{p}_m^m(t) - \frac{2t}{u^2} m\, \dot{\tilde{s}}_m^m \tilde{p}_m^m(t) + \frac{\left(m^2 - m \right) t^2 - m}{u^4} \tilde{s}_m^m \tilde{p}_m^m(t) \qquad (69)$$

Substituting equation (67) into equation (65) and re-arranging gives

$$\frac{\partial^2 V}{\partial t^2} = \frac{GM}{r} \sum_{m=0}^{N} \left\{ \begin{aligned} & \left(\ddot{\tilde{s}}_{m_c}^{m} \cos m\lambda + \ddot{\tilde{s}}_{m_s}^{m} \sin m\lambda \right) \\ & - \left(\dot{\tilde{s}}_{m_c}^{m} \cos m\lambda + \dot{\tilde{s}}_{m_s}^{m} \sin m\lambda \right) \frac{2t}{u^2} m \\ & + \left(\tilde{s}_{m_c}^{m} \cos m\lambda + \tilde{s}_{m_s}^{m} \sin m\lambda \right) \frac{\left(m^2 - m\right) t^2 - m}{u^4} \end{aligned} \right\} \tilde{p}_m^m(t) \tag{70}$$

where the sums $\ddot{\tilde{s}}_{m_c}^{m}$, $\ddot{\tilde{s}}_{m_s}^{m}$, $\dot{\tilde{s}}_{m_c}^{m}$, $\dot{\tilde{s}}_{m_s}^{m}$, $\tilde{s}_{m_c}^{m}$ and $\tilde{s}_{m_s}^{m}$ are calculated from a first application of Clenshaw's summation, using equations (61a), (67) and (36) with appropriate coefficients, and the derivative $\frac{\partial^2 V}{\partial t^2}$ evaluated by a second application of Clenshaw's summation.

DERIVATIVES OF THE POTENTIALS WITH RESPECT TO GEOCENTRIC CARTESIAN COORDINATES X,Y,Z AND LOCAL CARTESIAN COORDINATES x,y,z

The origin of the X,Y,Z coordinates lies at the "centre" of the reference ellipsoid, the same origin as the polar coordinates r,ψ,λ of the geopotential model. The *Z-axis* is coincident with the Earth's rotational axis, the *X-Z* plane is the Greenwich meridian plane (the origin of longitudes), the *X-Y* plane coincides with the Earth's equatorial plane (the origin of latitudes), the positive *X-axis* is in the direction of the intersection of the Greenwich meridian plane and the equatorial plane and the positive *Y-axis* is advanced 90° east along the equator.

The origin of the local Cartesian coordinates *x,y,z* lies at the point $P_0(r_0,\psi_0,\lambda_0)$. The positive *z-axis* is in the direction of increasing geocentric radius *r*, the *x-z* plane lies in the meridian plane passing through P_0 with the positive *x-axis* pointing north and the *x-y* plane is perpendicular to the *x-z* plane with the positive *y-axis* pointing east.

Geocentric and local Cartesian coordinates are related by the matrix equation (Tscherning 1976a, p. 74, eq. 8)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + C^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{71}$$

where

$$\begin{aligned} X_0 &= r_0 \cos\psi_0 \cos\lambda_0 \\ Y_0 &= r_0 \cos\psi_0 \sin\lambda_0 \\ Z_0 &= r_0 \sin\psi_0 \end{aligned} \quad \text{and} \quad C^T = \begin{bmatrix} -\sin\psi_0 \cos\lambda_0 & -\sin\lambda_0 & \cos\psi_0 \cos\lambda_0 \\ -\sin\psi_0 \sin\lambda_0 & \cos\lambda_0 & \cos\psi_0 \sin\lambda_0 \\ \cos\psi_0 & 0 & \sin\psi_0 \end{bmatrix}$$

Defining the vectors $X_i = \left[X_i \quad Y_i \quad Z_i\right]^T$ and $x_i = \left[x_i \quad y_i \quad z_i\right]^T$ Tscherning (1976a, pp. 74-75) shows that matrices of partial derivatives are related by

$$\left[\frac{\partial V}{\partial X_i}\right] = C^T \left[\frac{\partial V}{\partial x_i}\right] \tag{72a}$$

and

$$\left[\frac{\partial^2 V}{\partial X_i \partial X_j}\right] = C^T \left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right] C \tag{72b}$$

The vector of first derivatives is (Tscherning 1976a, p. 79, eq. 51)

$$\left[\frac{\partial V}{\partial x_i}\right] = \begin{bmatrix} \dfrac{u}{r}\dfrac{\partial V}{\partial t} \\ \dfrac{1}{ur}\dfrac{\partial V}{\partial \lambda} \\ \dfrac{\partial V}{\partial r} \end{bmatrix} \tag{73}$$

The symmetric matrix of second-order derivatives is (Tscherning 1976a, p. 80, eq. 58)

$$\left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right] = \begin{bmatrix} \frac{1}{r}\frac{\partial V}{\partial r} - \frac{t}{r^2}\frac{\partial V}{\partial t} + \frac{u^2}{r^2}\frac{\partial^2 V}{\partial t^2} & \cdots & \cdots \\ \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial t \partial \lambda} + \frac{t}{u^2}\frac{\partial V}{\partial \lambda}\right) & \frac{1}{r}\frac{\partial V}{\partial r} - \frac{t}{r^2}\frac{\partial V}{\partial t} + \frac{1}{u^2 r^2}\frac{\partial^2 V}{\partial \lambda^2} & \cdots \\ \frac{u}{r}\left(\frac{\partial^2 V}{\partial t \partial r} - \frac{1}{r}\frac{\partial V}{\partial t}\right) & \frac{1}{ur}\left(\frac{\partial^2 V}{\partial r \partial \lambda} - \frac{1}{r}\frac{\partial V}{\partial \lambda}\right) & \frac{\partial^2 V}{\partial r^2} \end{bmatrix} \tag{74}$$

Solving equation (72a) gives the first derivatives (Heiskanen & Moritz 1976, p. 230, eq. 6-18, noting that $\frac{\partial V}{\partial \psi} = \frac{\partial V}{\partial t}\frac{\partial t}{\partial \psi}$ where $\frac{\partial t}{\partial \psi} = \cos\psi = u$)

$$\frac{\partial V}{\partial X} = u\cos\lambda\frac{\partial V}{\partial r} - \frac{ut\cos\lambda}{r}\frac{\partial V}{\partial t} - \frac{\sin\lambda}{ur}\frac{\partial V}{\partial \lambda} \tag{75}$$

$$\frac{\partial V}{\partial Y} = u\sin\lambda\frac{\partial V}{\partial r} - \frac{ut\sin\lambda}{r}\frac{\partial V}{\partial t} + \frac{\cos\lambda}{ur}\frac{\partial V}{\partial \lambda} \tag{76}$$

$$\frac{\partial V}{\partial Z} = t\frac{\partial V}{\partial r} + \frac{u^2}{r}\frac{\partial V}{\partial t} \tag{77}$$

and solving equation (72b) gives the second-order derivatives (Tscherning 1976a, pp. 80-82, eqs 60-65)

$$\begin{aligned}\frac{\partial^2 V}{\partial X^2} = &\frac{u^2\cos^2\lambda}{r^2}\left(r^2\frac{\partial^2 V}{\partial r^2} + t^2\frac{\partial^2 V}{\partial t^2} - 2rt\frac{\partial^2 V}{\partial t \partial r}\right) + \frac{\sin^2\lambda}{r^2 u^2}\frac{\partial^2 V}{\partial \lambda^2} + \frac{1}{r}\left(1 - u^2\cos^2\lambda\right)\frac{\partial V}{\partial r} \\ &+ \frac{t}{r^2}\left(3u^2\cos^2\lambda - 1\right)\frac{\partial V}{\partial t} + \frac{2\cos\lambda\sin\lambda}{r^2}\left(\frac{1}{u^2}\frac{\partial V}{\partial \lambda} + t\frac{\partial^2 V}{\partial t \partial \lambda} - r\frac{\partial^2 V}{\partial r \partial \lambda}\right)\end{aligned} \tag{78}$$

$$\begin{aligned}\frac{\partial^2 V}{\partial Y^2} = &\frac{u^2\sin^2\lambda}{r^2}\left(r^2\frac{\partial^2 V}{\partial r^2} + t^2\frac{\partial^2 V}{\partial t^2} - 2rt\frac{\partial^2 V}{\partial t \partial r}\right) + \frac{\cos^2\lambda}{r^2 u^2}\frac{\partial^2 V}{\partial \lambda^2} + \frac{1}{r}\left(1 - u^2\sin^2\lambda\right)\frac{\partial V}{\partial r} \\ &+ \frac{t}{r^2}\left(3u^2\sin^2\lambda - 1\right)\frac{\partial V}{\partial t} - \frac{2\cos\lambda\sin\lambda}{r^2}\left(\frac{1}{u^2}\frac{\partial V}{\partial \lambda} + t\frac{\partial^2 V}{\partial t \partial \lambda} - r\frac{\partial^2 V}{\partial r \partial \lambda}\right)\end{aligned} \tag{79}$$

$$\frac{\partial^2 V}{\partial Z^2} = t^2\frac{\partial^2 V}{\partial r^2} + \frac{u^2}{r^2}\left(u^2\frac{\partial^2 V}{\partial t^2} + 2rt\frac{\partial^2 V}{\partial t \partial r} + r\frac{\partial V}{\partial r} - 3t\frac{\partial V}{\partial t}\right) \tag{80}$$

$$\frac{\partial^2 V}{\partial X \partial Y} = \frac{\sin\lambda\cos\lambda}{r^2}\left\{u^2\left[t^2\frac{\partial^2 V}{\partial t^2} + r^2\frac{\partial^2 V}{\partial r^2} - r\frac{\partial V}{\partial r} + t\left(3\frac{\partial V}{\partial t} - 2r\frac{\partial^2 V}{\partial t \partial r}\right)\right] - \frac{1}{u^2}\frac{\partial^2 V}{\partial \lambda^2}\right\}$$
$$+ \frac{\sin^2\lambda - \cos^2\lambda}{r^2}\left(t\frac{\partial^2 V}{\partial t \partial \lambda} + \frac{1}{u^2}\frac{\partial V}{\partial \lambda} - r\frac{\partial^2 V}{\partial r \partial \lambda}\right) \quad (81)$$

$$\frac{\partial^2 V}{\partial X \partial Z} = \frac{\cos\lambda}{r^2}\left\{ut\left(-u^2\frac{\partial^2 V}{\partial t^2} + r^2\frac{\partial^2 V}{\partial r^2} - r\frac{\partial V}{\partial r}\right) + u\left(\left[3t^2 - 1\right]\frac{\partial V}{\partial t} + r\left[u^2 - t^2\right]\frac{\partial^2 V}{\partial t \partial r}\right)\right\}$$
$$- \frac{\sin\lambda}{r^2}\left(u\frac{\partial^2 V}{\partial t \partial \lambda} + \frac{rt}{u}\frac{\partial^2 V}{\partial r \partial \lambda}\right) \quad (82)$$

$$\frac{\partial^2 V}{\partial Y \partial Z} = \frac{\sin\lambda}{r^2}\left\{ut\left(-u^2\frac{\partial^2 V}{\partial t^2} + r^2\frac{\partial^2 V}{\partial r^2} - r\frac{\partial V}{\partial r}\right) + u\left(\left[3t^2 - 1\right]\frac{\partial V}{\partial t} + r\left[u^2 - t^2\right]\frac{\partial^2 V}{\partial t \partial r}\right)\right\}$$
$$+ \frac{\cos\lambda}{r^2}\left(u\frac{\partial^2 V}{\partial t \partial \lambda} + \frac{rt}{u}\frac{\partial^2 V}{\partial r \partial \lambda}\right) \quad (83)$$

In equations (72) to (83) V can be replaced by W or T, also by U or V_U if derivatives of the normal gravity potential or normal gravitational potential are required. In the latter cases, derivatives with respect to λ will be zero since the normal ellipsoid is rotationally symmetric.

Equations (75) to (83) give explicit relationships between geocentric Cartesian functions on the one side and geocentric spherical functions on the other. Identical numerical results can be obtained from the matrix equations (72) when values for the spherical functions are stored as (i) a 3×1 vector as per equation (73), (ii) a 3×3 symmetric matrix as per equation (74) and (iii) a 3×3 orthogonal coefficient matrix C as per equation (71). Tscherning (1976a, pp. 82-84) shows that with an appropriate orthogonal coefficient matrix, this technique can be used to determine derivatives of the normal gravity potential U with respect to other Cartesian reference frames, such as a local system with the *z-axis* coincident with the direction of the normal gravity vector.

It should be noted that some spherical functions have the variable u in the denominator, since $u = \cos\psi$, these functions will be indeterminate at the north or south poles and computer algorithms should be constructed so as to guard against this event.

When developing the equation for the disturbing potential T [equation (11)] it was assumed that the parameters GM and a $(q = a/r)$ had the same values in the equations for the gravitational potentials V_W and V_U [equations (4) and (10) respectively] and also that ω had the same value for the Earth and the normal ellipsoid. This may not be the case in practice, in which event, equation (11) should be modified to

$$T(r,\psi,\lambda) = \frac{GM_W}{r} \sum_{m=0}^{N} \sum_{n=m}^{N} \left\{ \left(C_n^m - xy\,A_n^m \right) \cos m\lambda + S_n^m \sin m\lambda \right\} q_W^n \, P_n^m(t) + \left(R_W - R_U \right)$$

where $x = \dfrac{GM_U}{GM_W}$, $y = \dfrac{q_U}{q_W} = \dfrac{a_U}{a_W}$ and the subscripts W and U refer to values applicable to the Earth and the normal ellipsoid respectively. R_W and R_U are given by equation (2) with appropriate values for ω. Derivatives of T (using the equation above) are obtained in the same way as derivatives of W.

NUMERICAL CHECKS ON DERIVATIVES

Derivatives of V and W must satisfy the following equations, firstly Laplace's equation (Cartesian and polar form)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{84a}$$

$$\frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} - \frac{2t}{r^2}\frac{\partial V}{\partial t} + \frac{1}{u^2 r^2}\frac{\partial^2 V}{\partial \lambda^2} + \frac{u^2}{r^2}\frac{\partial^2 V}{\partial t^2} = 0 \tag{84b}$$

[Equation (84b) is the trace (sum of the diagonal elements) of the matrix equation (74), given by Tscherning (1976a, p. 81, eq. 59) and also in a similar form by Heiskanen and Moritz (1967, p. 19, eq. 1-41)] and secondly, for a point on or above the Earth's surface

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = \frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} + \frac{\partial^2 W}{\partial Z^2} = 2\omega^2 \tag{85}$$

For points on the reference ellipsoid (GRS80), derivatives of the normal gravity potential U with respect to spherical coordinates r,t,λ and geocentric Cartesian coordinates X,Y,Z satisfy equations for the normal gravity γ and the geodetic latitude ϕ (Tscherning 1976a, eqs 68 & 69, p. 83). Also, derivatives of U, with respect to a local Cartesian system with the *z-axis* in the direction of the normal gravity vector, satisfy equations arising from the rotational symmetry of the normal ellipsoid (Tscherning 1976a, p. 84).

SUMMARY

Clenshaw's summation is an efficient numerical technique for calculating potentials of the Earth and their derivatives. The method, developed by C.W. Clenshaw in 1955 for the summation of Chebyshev polynomials, has the advantage that truncation errors (inherent in the evaluation of coefficients of recurrence relationships when N is large) do not accumulate to cause significant error in the sum of a series of orthogonal polynomials. The numerical accuracy of the technique (applied to geodetic problems) has been verified by Gleeson (1985) who compared the method with traditional analytic techniques involving the recursive computation of Legendre polynomials. Gleeson also showed that the technique offers considerable savings in computation time (compared with analytic methods) verifying the studies by Tscherning, Rapp and Goad (1983).

This paper has presented the necessary equations for the computation of the Earth's gravity potential W and its derivatives as well as a computer algorithm demonstrating Clenshaw's summation. Interested readers should be able to modify the equations to enable the computation of potentials U and T and their derivatives.

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Note: A computer program for the computation of the Earth's potentials and derivatives is available from the author. e-mail address: deakin@rmit.edu.au

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APPENDIX A C computer algorithm for calculating the Earth's gravity potential *W.*

The algorithm requires *GM* the product of the Earth's gravitational constant *G* and its mass *M*, *Omega* the angular velocity of the Earth, *a* the equatorial radius of the Earth, *X,Y,Z* geocentric Cartesian coordinates of a point and a set of quasi-normalised geopotential coefficients stored in an array *Cvec[]*. The algorithm does not include header files, normally required in C programs, proper definitions of integer and real variables, or the definition of the real array *Cvec[]*. No method of input or output is given.

```
Cvec[0] = 1.0;                                    Cvec[] is an array containing quasi-normalised
                                                  geopotential coefficients C(n,m) and S(n,m) where n is
                                                  degree and m is order. The order of elements in
                                                  Cvec[] is C(0,0), C(1,0), C(1,1), S(1,1), C(2,0), C(2,1),
                                                  S(2,1), C(2,2), S(2,2), C(3,0), C(3,1), S(3,1), C(3,2),
                                                  S(3,2), ... , C(N,N), S(N,N) where N is the maximum
                                                  degree and order. Cvec[] does not contain the
                                                  coefficients S(n,m=0) since these coefficients are zero.
lambda = atan2(Y,X);                              lambda is longitude
Om2  = Omega*Omega                                Omega is the angular velocity of the Earth.
r    = sqrt(X*X + Y*Y + Z*Z);                     r is geocentric radius; X,Y,Z are geocentric coords.
p    = sqrt(X*X + Y*Y);                           p is perpendicular distance from Z axis
p2   = p*p;
t    = Z/r;                                       t = sin(psi) where psi is geocentric latitude
u    = p/r;                                       u = cos(psi)
q    = a/r                                        a is semi-major axis of reference ellipsoid
q2   = q*q
V    = 0.0;                                       initialise gravitational potential V
km   = (N+1)*(N+1);                               (km - 1) is pointer to coefficient S(N,N) in Cvec[]
for(m=N; m>=0; m--) {                             outer loop "m" in potential summation, m is order
     Sc = Ss = Sc1 = Ss1 = 0.0;                   initialise variables for Clenshaw recursion
     km = km - ((m==0)? 1 : 2);                   km = km - 1 if m = 0; else km = km - 2
     k  = km;                                     set k to point at coefficient C(n,m) in Cvec[]
     for(n=N; n>=m; n--) {                        inner loop "n" in potential summation, n is degree
          x   = sqrt(n+m+1)*sqrt(n-m+1);
          y   = sqrt(n+m+2)*sqrt(n-m+2);
          a1  = q*(n+n+1)/x;                      set coefficient a(n+1) in 1st Clenshaw summation
          a1t = a1*t;
          b2  = -q2*x/y;                          set coefficient b(n+2) in 1st Clenshaw summation
          Cnm = Cvec[k];                          Cnm, Snm are coefficients C(n,m), S(n,m) in Cvec[]
          Snm = ((m==0)? 0 : Cvec[k+1]);          Snm = 0 when m = 0
          Sc2 = Sc1;                              Clenshaw's recursive summation for Sc
          Sc1 = Sc;
          Sc  = a1t*Sc1 + b2*Sc2 + Cnm;
          Ss2 = Ss1;                              Clenshaw's recursive summation for Ss
          Ss1 = Ss;
          Ss  = a1t*Ss1 + b2*Ss2 + Snm;
          k   = k - (n+n-1);
     }                                            end of "n" loop in potential summation
     x   = sqrt(m+m+1);
     y   = sqrt(m+m+2);
     a1  = q*x/y;                                 coefficient a(n+1) in 2nd Clenshaw summation.
     a1u = a1*u;
     V   = a1u*V + Sc*cos(m*lambda)
                 + Ss*sin(m*lambda);              2nd Clenshaw recursive summation
}                                                 end of "m" loop and potential summation
V = GM/r*V;                                       V is Earth's gravitational potential
R = Om2*p2/2.0;                                   R is rotational potential
W = V + R;                                        W is Earth's gravity potential
```

This computer algorithm closely follows those of Tscherning and Poder (1981) and Tscherning, Rapp and Goad (1983).

For the purposes of checking a computer program, similar to that above, the following constants and geopotential coefficients are useful.

For the Geodetic Reference System 1980 (GRS80)

```
Omega = 7.292115E-05 rad/s
GM    = 3.986005E+14 m3/s2
a     = 6378137.000 m
```

Fully-normalised geopotential coefficients (OSU91A1F) to degree and order 4

n	m	C(n,m)	S(n,m)
0	0	0.000000000000E+00	0.000000000000E+00
1	0	0.000000000000E+00	0.000000000000E+00
1	1	0.000000000000E+00	0.000000000000E+00
2	0	-0.484165532804E-03	0.000000000000E+00
2	1	0.857179552165E-12	0.289607376372E-11
2	2	0.243815798120E-05	-0.139990174643E-05
3	0	0.957139401177E-06	0.000000000000E+00
3	1	0.202968777310E-05	0.249431310090E-06
3	2	0.904648670700E-06	-0.620437816800E-06
3	3	0.720295507400E-06	0.141470959443E-05
4	0	0.540441629840E-06	0.000000000000E+00
4	1	-0.535373285210E-06	-0.474065010407E-06
4	2	0.350729847400E-06	0.663967363224E-06
4	3	0.991080200230E-06	-0.202148896490E-06
4	4	-0.190576531700E-06	0.309704028950E-06

Quasi-normalised geopotential coefficients (OSU91A1F) to degree and order 4

n	m	C(n,m)	S(n,m)
0	0	0.000000000000E+00	0.000000000000E+00
1	0	0.000000000000E+00	0.000000000000E+00
1	1	0.000000000000E+00	0.000000000000E+00
2	0	-0.108262704371E-02	0.000000000000E+00
2	1	0.271063974856E-11	0.915818936521E-11
2	2	0.771013251591E-05	-0.442687801917E-05
3	0	0.253235282554E-05	0.000000000000E+00
3	1	0.759439624906E-05	0.933286503891E-06
3	2	0.338488538116E-05	-0.232146574026E-05
3	3	0.269509900592E-05	0.529335860414E-05
4	0	0.162132488952E-05	0.000000000000E+00
4	1	-0.227139648263E-05	-0.201128750149E-05
4	2	0.148802072077E-05	0.281697495013E-05
4	3	0.420479718169E-05	-0.857645133105E-06
4	4	-0.808547747400E-06	0.131396291419E-05

Geocentric coordinates (latitude S37° 48′ and longitude E144° 58′ on surface of GRS80 ellipsoid).

```
X = -4131810.563 m
Y =  2896708.708 m
Z = -3887927.165 m
```

Results: V = 62569226.824976,   R = 67699.091078, W = 62636925.916054 m2/s2